Separation of variables in the hydrodynamic stability equations

2006 J. Phys. A: Math. Gen. 39 7141

(http://iopscience.iop.org/0305-4470/39/22/024)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 7141–7160 doi:10.1088/0305-4470/39/22/024

Separation of variables in the hydrodynamic stability equations

Alexander Zhalij[1], **Georgy I Burde**[2] **and Ildar Sh Nasibullayev**[2,3]

[1] Institute of Mathematics of the Academy of Sciences of Ukraine, Tereshchenkivska Street 3, 01601 Kyiv-4, Ukraine
[2] Jacob Blaustein Institute for Desert Research, Ben-Gurion University, Sede-Boker Campus, 84990, Israel
[3] Institute of Mechanics, Russian Academy of Sciences, 450054 Ufa, Russia

E-mail: zhaliy@imath.kiev.ua, georg@bgu.ac.il and ildar@bgu.ac.il

Received 9 June 2005, in final form 16 March 2006
Published 16 May 2006
Online at stacks.iop.org/JPhysA/39/7141

Abstract
The problem of variable separation in the linear stability equations, which govern the disturbance behaviour in viscous incompressible fluid flows, is discussed. The so-called direct approach, in which a form of the 'ansatz' for a solution with separated variables as well as a form of reduced ODEs are postulated from the beginning, is applied. The results of application of the method are the new coordinate systems and the most general forms of basic flows, which permit the postulated form of separation of variables. Then the basic flows are specified by the requirement that they themselves satisfy the Navier–Stokes equations. Calculations are made for the (1+3)-dimensional disturbance equations written in Cartesian and cylindrical coordinates. The fluid dynamics interpretation and stability properties of some classes of the exact solutions of the Navier–Stokes equations, defined as flows for which the stability analysis can be reduced via separation of variables to the eigenvalue problems of ordinary differential equations, are discussed. The eigenvalue problems are solved numerically with the help of the spectral collocation method based on Chebyshev polynomials. For some classes of perturbations, the eigenvalue problems can be solved analytically. Those unique examples of exact (explicit) solution of the nonparallel unsteady flow stability problems provide a very useful test for numerical methods of solution of eigenvalue problems, and for methods used in the hydrodynamic stability theory, in general.

PACS numbers: 02.30.Ik, 04.20.Jb, 47.15.Fe, 47.20.Gv, 02.60.Lj

1. Introduction

Problems of hydrodynamic stability are of great theoretical and practical interest, as evidenced by the number of publications devoted to this subject. The linear stability theory (see, e.g., Drazin and Reid (1981)) for a particular flow starts with a solution of the equations of motion representing this basic flow. One then considers this solution with a small perturbation superimposed. Substituting the perturbed solution into the equations of motion and neglecting all terms that involve the square of the perturbation amplitude yield the linear stability equations which govern the behaviour of the perturbation. If the perturbation dies away the original flow is said to be stable, and if the perturbation grows, the flow is said to be unstable. Whether a small disturbance that is superimposed upon a known primary flow will be amplified or damped depends on the pattern of the primary flow and the nature of the disturbance. The linearization provides a means of allowing for the many different forms that the disturbance can take. In the method of normal modes, small disturbances are resolved into modes, which may be treated separately because each satisfies the linear equations and there are no interactions between different modes.

Thus, the mathematical problem of the determination of stability of a given flow involves deriving a set of perturbation equations obtained from the Navier–Stokes equations by linearization around this basic flow and finding a set of possible solutions which would permit splitting a perturbation into normal modes. For a steady-state basic flow, normal modes depending on time exponentially, with a complex exponent λ, are permissible—the sign of the real part of λ indicates whether the disturbance grows or decays in time. If further separation of variables is possible, it makes the stability problem amenable to the normal mode analysis in its common form when the problem reduces to that of solving a set of ordinary differential equations. It can be done, however, only for basic flows of specific forms—mostly those are the parallel flows or their axial symmetric counterparts. With the parallel flow assumption, separation of variables in the governing stability equations leads to the fourth-order ordinary differential equation, the Orr–Sommerfeld equation. When this equation is solved with proper boundary conditions, the problem of linear stability of parallel flows is reduced to a 2-point boundary (eigen) value problem.

For nonparallel basic flows, when the coefficients in the equations for disturbance flow are dependent not only on the normal to the flow coordinate but also on the other coordinates, the corresponding operator does not separate unless certain terms are ignored. The approximation, which neglects the nonparallel terms and relates the stability characteristics to those of the equivalent parallel flow has been extensively used for the stability analysis of the so-called 'weakly nonparallel' flows to retain the great advantage of reducing the disturbance equations to ordinary differential equations (see review by Reed and Saric (1996)). Also a number of approximate approaches, which seek to account for effects of non-parallelism through equations including higher-order terms than those in the Orr–Sommerfeld equation, have been developed (see reviews in Reed and Saric (1996), Herbert (1997), Fazel and Konzelmann (1990), Govindarajan and Narishama (1999), Saric *et al* (2003)).

Laminar flows, that are observed in nature and laboratory, are never perfectly steady and are often very unsteady. If the basic flow is non-steady, this brings about great difficulties in theoretical studies of the instability since the normal modes containing an exponential time factor $\exp(\lambda t)$ are not applicable here (some success has been achieved in studying stability of the time-periodic basic states when Floquet theory can be applied—see Drazin and Reid (1981)). If an unsteady flow is non-parallel, it should further complicate matters. Therefore stability of viscous incompressible flows developing both in space and time is a little studied topic in the theory of hydrodynamic stability. As a matter of fact, there are no examples of the unsteady non-parallel solutions of the incompressible Navier–Stokes equations, for which the corresponding linear stability problem is exactly solvable via separation of variables.

All the above said shows that the method of separation of variables is of a fundamental importance for the hydrodynamic stability problems. Stability analysis based on this method describes local behaviour or development of small disturbances with an eigensolution of ordinary differential equations and therefore is free from the initial conditions under which the disturbance is introduced into the flow. Reducing the disturbance equations from partial differential equations to ordinary differential equations allows quick numerical solution. The importance of the method of separation of variables is even more enhanced in the case of time-dependent basic states when the method of normal modes is not applicable at all. Till now, the method of separation of variables has been used for stability analysis in an intuitive way which makes it generally applicable only to the stability problems of the steady-state parallel flows.

Recently, the so-called direct approach to separation of variables in linear PDEs has been developed by a proper formalizing of the features of the notion of separation of variables (see, e.g., Zhdanov and Zhalij (1999a, 1999b)). In this approach, a form of the 'Ansatz' for a solution with separated variables as well as a form of reduced ODEs, which should be obtained as a result of the variable separation, are postulated from the beginning. The method has been successfully applied to several equations of mathematical physics (see, e.g., Zhalij (1999), Zhdanov and Zhalij (1999a, 1999b), Zhalij (2002)).

In the present paper, we apply this approach to the linear stability equations which govern the disturbance behaviour in viscous incompressible fluid flows. The calculations are made for the linear stability equations written in Cartesian and cylindrical coordinates. The forms of the Ansatz for the solutions with separated variables are chosen such that the perturbations written in a new non-stationary coordinate system were periodic in two space-like coordinates. The new coordinate systems and the most general forms of basic flows, which permit the postulated form of separation of variables in the equations for disturbances, are determined as the result of application of the method. Then the basic flows are specified by the requirement that they exactly satisfy the Navier–Stokes equations.

Further we discuss the fluid dynamics interpretation and stability properties of some classes of the exact solutions of the Navier–Stokes equations defined as flows for which the stability analysis can be reduced via separation of variables to the eigenvalue problems of ordinary differential equations. The eigenvalue problems were solved numerically with the help of the spectral collocation method based on Chebyshev polynomials. It appears that for some classes of perturbations, the eigenvalue problems can be solved analytically. Those unique examples of exact (even explicit) solution of the nonparallel unsteady flow stability problems provide a very useful test for numerical methods of solution of eigenvalue problems, and for methods used in the hydrodynamic stability theory, in general.

2. Separation of variables in Cartesian coordinates

Throughout the paper we deal with the Navier–Stokes equations governing flows of incompressible Newtonian fluids. Then the equation of motion and the equation of continuity are

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} + (\hat{\mathbf{v}}\nabla)\,\hat{\mathbf{v}} = -\frac{1}{\rho}\nabla \hat{p} + \nu\nabla^2\hat{\mathbf{v}} \qquad \text{and} \qquad \nabla\hat{\mathbf{v}} = 0, \tag{1}$$

where ρ is the constant density and ν is the constant kinematic viscosity of the fluid.

As usual in the stability analysis, we split the velocity and pressure fields $(\hat{v}_x, \hat{v}_y, \hat{v}_z, \hat{p})$ into two problems: the basic flow problem (V_x, V_y, V_z, P) and a perturbation one (v_x, v_y, v_z, p),

$$\hat{v}_x = V_x + v_x, \qquad \hat{v}_y = V_y + v_y, \qquad \hat{v}_z = V_z + v_z, \qquad \hat{p} = P + p. \tag{2}$$

Introducing (2) into the Navier–Stokes equations (1) and neglecting all terms that involve the square of the perturbation amplitude, while imposing the requirement that the basic flow variables (V_x, V_y, V_z, P) themselves satisfy the Navier–Stokes equations, one arrives at the following set of linear stability equations in the Cartesian coordinates:

$$\begin{aligned}
&\frac{\partial v_x}{\partial t} + V_x\frac{\partial v_x}{\partial x} + v_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial v_x}{\partial y} + v_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial v_x}{\partial z} + v_z\frac{\partial V_x}{\partial z} \\
&\qquad = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right), \\
&\frac{\partial v_y}{\partial t} + V_x\frac{\partial v_y}{\partial x} + v_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial v_y}{\partial y} + v_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial v_y}{\partial z} + v_z\frac{\partial V_y}{\partial z} \\
&\qquad = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right), \\
&\frac{\partial v_z}{\partial t} + V_x\frac{\partial v_z}{\partial x} + v_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial v_z}{\partial y} + v_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial v_z}{\partial z} + v_z\frac{\partial V_z}{\partial z} \\
&\qquad = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right), \\
&\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0.
\end{aligned} \tag{3}$$

The system (3) has variable coefficients dependent on the basic flow velocity fields V_x, V_y, V_z so that one cannot apply the standard Fourier transformation or classical normal mode approach. A regular way for solving (3) is the method of separation of variables but, in this respect, natural questions arise: which basic flows V_x, V_y, V_z allow separability of the system (3) and what is the form of separable solutions? One of the principal objectives of the present paper is to provide an efficient algorithm for answering these kinds of questions.

There are three important elements of the method of separation of variables for partial differential equations. The first one is a choice of a special *Ansatz* for a class of particular solutions to be found in a separated (factorized) form via several functions of one variable. The second element is a requirement to have as an output of the whole procedure several ordinary differential equations. The third important point is that the thus obtained solutions depend on auxiliary parameters (separation constants). The principal idea of the approach to separation of variables in partial differential equations suggested in Zhdanov and Zhalij (1999a), Zhalij (2002) was to postulate these three features in order to develop a regular procedure for obtaining solutions with separated variables.

Let us introduce a new coordinate system $t, \xi = \xi(t, x), \eta = \eta(t, y), \gamma = \gamma(t, z)$. First, we choose the Ansatz for a solution to be found

$$\begin{aligned}
v_x &= T(t)\exp(a\xi + s\gamma + mS(t))h(\eta), \\
v_y &= T(t)\exp(a\xi + s\gamma + mS(t))f(\eta), \\
v_z &= T(t)\exp(a\xi + s\gamma + mS(t))g(\eta), \\
p &= T_1(t)\exp(a\xi + s\gamma + mS(t))k(\eta),
\end{aligned} \tag{4}$$

where functions $T(t), T_1(t), S(t), \xi(t,x), \eta(t,y), \gamma(t,z)$ are not fixed *a priori* but chosen in such a way that inserting the expressions (4) into system of PDEs (3) yields a system of three second-order and one first-order ordinary differential equations for four functions $h(\eta), f(\eta), g(\eta), k(\eta)$. To get constraints on functions $T, T_1, S, \xi, \eta, \gamma$ we formalize a reduction procedure as follows.

First, we postulate the form of the resulting system of ordinary differential equations as follows:

$$\begin{aligned}
h''(\eta) &= U_{11}g'(\eta) + U_{12}h'(\eta) + U_{13}k'(\eta) + U_{14}f(\eta) + U_{15}g(\eta) + U_{16}h(\eta) + U_{17}k(\eta),\\
f''(\eta) &= U_{21}g'(\eta) + U_{22}h'(\eta) + U_{23}k'(\eta) + U_{24}f(\eta) + U_{25}g(\eta) + U_{26}h(\eta) + U_{27}k(\eta),\\
g''(\eta) &= U_{31}g'(\eta) + U_{32}h'(\eta) + U_{33}k'(\eta) + U_{34}f(\eta) + U_{35}g(\eta) + U_{36}h(\eta) + U_{37}k(\eta),\\
f'(\eta) &= U_{41}f(\eta) + U_{42}g(\eta) + U_{43}h(\eta) + U_{44}k(\eta).
\end{aligned} \tag{5}$$

Here U_{ij} are second-order polynomials with respect to spectral parameters a, s, m with coefficients, which are some smooth functions on η and should be determined on the next steps of the algorithm. Next, we insert the expressions (4) into (3) which yields a system of PDEs containing the functions ξ, η, γ and their first- and second-order partial derivatives, and the functions $f(\eta), g(\eta), k(\eta)$ and their derivatives. Further we replace the derivatives $h''(\eta), f''(\eta), g''(\eta), f'(\eta)$ by the corresponding expressions from the right-hand sides of (5).

Now we regard $h'(\eta), g'(\eta), k'(\eta), h(\eta), f(\eta), g(\eta), k(\eta)$ as the new independent variables. As the functions $\xi(x,t), \eta(y,t), \gamma(z,t), T(t), T_1(t), S(t)$, basic flows V_x, V_y, V_z and coefficients of the polynomials U_{ij} (which are functions of η) are independent of these variables, we can require that the obtained equality is transformed into identity under arbitrary $h'(\eta), g'(\eta), k'(\eta), h(\eta), f(\eta), g(\eta), k(\eta)$. In other words, we should split the equality with respect to these variables. After splitting we get an overdetermined system of nonlinear partial differential equations for unknown functions $\xi(x,t), \eta(y,t), \gamma(z,t), T(t), T_1(t), S(t)$, basic flows V_x, V_y, V_z and coefficients of the polynomials U_{ij}. At the last step we solve the above system to get an exhaustive description of coordinate systems providing separability of equations (3), as well as all possible basic flows V_x, V_y, V_z such that the system (3) is solvable by the method of separation of variables.

Thus, the problem of variable separation in equation (3) reduces to integrating the overdetermined system of PDEs for unknown functions $\xi(x,t), \eta(y,t), \gamma(z,t), T(t), T_1(t), S(t)$, basic flows V_x, V_y, V_z and coefficients of the polynomials U_{ij}. This has been done with the aid of *Mathematica* package.

Below we list the results.

The most general form of the basic flow is:

$$\begin{aligned}
V_x &= \nu A(\eta)T(t) - \frac{c_1'(t) + xT'(t)}{T(t)},\\
V_y &= \nu B(\eta)T(t) - \frac{c_2'(t) + yT'(t)}{T(t)},\\
V_z &= \nu C(\eta)T(t) - \frac{c_3'(t) + zT'(t)}{T(t)}.
\end{aligned}$$

The forms of the perturbations v_x, v_y, v_z and p are:

$$\begin{aligned}
v_x &= T(t)\exp\left(a\xi + s\gamma + m\int T(t)^2\,\mathrm{d}t\right)h(\eta),\\
v_y &= T(t)\exp\left(a\xi + s\gamma + m\int T(t)^2\,\mathrm{d}t\right)f(\eta),
\end{aligned}$$

$$v_z = T(t)\exp\left(a\xi + s\gamma + m\int T(t)^2\,\mathrm{d}t\right)g(\eta),$$
$$p = \rho T(t)^2\exp\left(a\xi + s\gamma + m\int T(t)^2\,\mathrm{d}t\right)k(\eta),$$

where

$$\xi = T(t)x + c_1(t), \qquad \eta = T(t)y + c_2(t), \qquad \gamma = T(t)z + c_3(t).$$

The equations with separated variables are

$$(m - a^2\nu - s^2\nu + a\nu A(\eta) + s\nu C(\eta))h(\eta) + ak(\eta) + \nu(f(\eta)A'(\eta) + B(\eta)h'(\eta) - h''(\eta)) = 0,$$
$$f(\eta)(m - a^2\nu - s^2\nu + a\nu A(\eta) + s\nu C(\eta) + \nu B'(\eta)) + \nu B(\eta)f'(\eta) + k'(\eta) - \nu f''(\eta) = 0,$$
$$(m - a^2\nu - s^2\nu + a\nu A(\eta) + s\nu C(\eta))g(\eta) + sk(\eta) + \nu(f(\eta)C'(\eta) + B(\eta)g'(\eta) - g''(\eta)) = 0,$$
$$sg(\eta) + ah(\eta) + f'(\eta) = 0.$$

The restrictions on the forms of the basic flows following from the requirement that they themselves satisfy the Navier–Stokes equations lead to the two following cases:

Case I

$$\xi = \frac{1}{\sqrt{t}}x + c_1(t), \qquad \eta = \frac{1}{\sqrt{t}}y + c_2(t), \qquad \gamma = \frac{1}{\sqrt{t}}z + c_3(t); \tag{6}$$

$$\begin{aligned} V_x &= \frac{x}{2t} + \nu A(\eta)\frac{1}{\sqrt{t}} - c_1'(t)\sqrt{t},\\ V_y &= -\frac{y}{t} - \frac{1}{\sqrt{t}}\left(tc_2'(t) + \frac{3}{2}c_2(t)\right),\\ V_z &= \frac{z}{2t} + \nu C(\eta)\frac{1}{\sqrt{t}} - c_3'(t)\sqrt{t}, \end{aligned} \tag{7}$$

and the functions $A(x)$ and $C(x)$ satisfy the equations

$$3A'(\eta) + 3\eta A''(\eta) + 2\nu A'''(\eta) = 0, \tag{8}$$
$$3C'(\eta) + 3\eta C''(\eta) + 2\nu C'''(\eta) = 0, \tag{9}$$

which can be solved in terms of the error functions and the generalized hypergeometric functions (see section 4).

For this case the separation Ansatz takes the form

$$v_x = t^s\,\mathrm{e}^{a\xi + m\gamma}f(\eta), \tag{10}$$
$$v_y = t^s\,\mathrm{e}^{a\xi + m\gamma}g(\eta), \tag{11}$$
$$v_z = t^s\,\mathrm{e}^{a\xi + m\gamma}h(\eta), \tag{12}$$
$$p = \rho t^{s-1/2}\,\mathrm{e}^{a\xi + m\gamma}\pi(\eta). \tag{13}$$

Case II

$$\xi = x + c_1(t), \qquad \eta = y + c_2(t), \qquad \gamma = z + c_3(t);$$
$$V_x = A_1\eta^2 + A_2\eta - c_1'(t), \qquad V_y = -c_2'(t), \qquad V_z = C_1\eta^2 + C_2\eta - c_3'(t).$$

For this case the separation Ansatz is

$$v_x = \mathrm{e}^{a\xi + s\gamma + mt}f(\eta), \qquad v_y = \mathrm{e}^{a\xi + s\gamma + mt}g(\eta),$$
$$v_z = \mathrm{e}^{a\xi + s\gamma + mt}h(\eta), \qquad p = \rho\,\mathrm{e}^{a\xi + s\gamma + mt}\pi(\eta).$$

3. Separation of variables in cylindrical coordinates

The Navier–Stokes equations in cylindrical coordinates (r, φ, z) are:

$$\begin{aligned}
&\frac{\partial \hat{v}_r}{\partial t} + \hat{v}_r \frac{\partial \hat{v}_r}{\partial r} + \frac{\hat{v}_\varphi}{r}\frac{\partial \hat{v}_r}{\partial \varphi} - \frac{\hat{v}_\varphi^2}{r} + \hat{v}_z \frac{\partial \hat{v}_r}{\partial z} \\
&\qquad = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial r} + \nu\left(\frac{\partial^2 \hat{v}_r}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{v}_r}{\partial r} - \frac{\hat{v}_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 \hat{v}_r}{\partial \varphi^2} - \frac{2}{r^2}\frac{\partial \hat{v}_\varphi}{\partial \varphi} + \frac{\partial^2 \hat{v}_r}{\partial z^2}\right), \\
&\frac{\partial \hat{v}_\varphi}{\partial t} + \hat{v}_r \frac{\partial \hat{v}_\varphi}{\partial r} + \frac{\hat{v}_\varphi}{r}\frac{\partial \hat{v}_\varphi}{\partial \varphi} + \frac{\hat{v}_r \hat{v}_\varphi}{r} + \hat{v}_z \frac{\partial \hat{v}_\varphi}{\partial z} \\
&\qquad = -\frac{1}{\rho r}\frac{\partial \hat{p}}{\partial \varphi} + \nu\left(\frac{\partial^2 \hat{v}_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{v}_\varphi}{\partial r} - \frac{\hat{v}_\varphi}{r^2} + \frac{1}{r^2}\frac{\partial^2 \hat{v}_\varphi}{\partial \varphi^2} + \frac{2}{r^2}\frac{\partial \hat{v}_r}{\partial \varphi} + \frac{\partial^2 \hat{v}_\varphi}{\partial z^2}\right), \\
&\frac{\partial \hat{v}_z}{\partial t} + \hat{v}_r \frac{\partial \hat{v}_z}{\partial r} + \frac{\hat{v}_\varphi}{r}\frac{\partial \hat{v}_z}{\partial \varphi} + \hat{v}_z \frac{\partial \hat{v}_z}{\partial z} \\
&\qquad = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial z} + \nu\left(\frac{\partial^2 \hat{v}_z}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{v}_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \hat{v}_z}{\partial \varphi^2} + \frac{\partial^2 \hat{v}_z}{\partial z^2}\right), \\
&\frac{\partial \hat{v}_r}{\partial r} + \frac{\hat{v}_r}{r} + \frac{1}{r}\frac{\partial \hat{v}_\varphi}{\partial \varphi} + \frac{\partial \hat{v}_z}{\partial z} = 0,
\end{aligned} \tag{14}$$

where $\hat{v}_r, \hat{v}_\varphi, \hat{v}_z, \hat{p}$ are the velocity and pressure fields. We split them into the basic flow and perturbation parts

$$\hat{v}_r = V_r + v_r, \qquad \hat{v}_\varphi = V_\varphi + v_\varphi, \qquad \hat{v}_z = V_z + v_z, \qquad \hat{p} = P + p, \tag{15}$$

where V_r, V_φ, V_z, P are the basic flow fields and v_r, v_φ, v_z, p are the perturbations. Substituting (2) into (14) with subsequent linearization with respect to the perturbations yields

$$\begin{aligned}
&\frac{\partial v_r}{\partial t} + V_r\frac{\partial v_r}{\partial r} + v_r\frac{\partial V_r}{\partial r} + \frac{V_\varphi}{r}\frac{\partial v_r}{\partial \varphi} + \frac{v_\varphi}{r}\frac{\partial V_r}{\partial \varphi} + V_z\frac{\partial v_r}{\partial z} + v_z\frac{\partial V_r}{\partial z} - 2\frac{V_\varphi v_\varphi}{r} \\
&\qquad = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \varphi^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi} - \frac{v_r}{r^2}\right), \\
&\frac{\partial v_\varphi}{\partial t} + V_r\frac{\partial v_\varphi}{\partial r} + v_r\frac{\partial V_\varphi}{\partial r} + \frac{V_\varphi}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_\varphi}{r}\frac{\partial V_\varphi}{\partial \varphi} + V_z\frac{\partial v_\varphi}{\partial z} + v_z\frac{\partial V_\varphi}{\partial z} + \frac{V_r v_\varphi}{r} + \frac{v_r V_\varphi}{r} \\
&\qquad = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{\partial^2 v_\varphi}{\partial z^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r^2}\right), \\
&\frac{\partial v_z}{\partial t} + V_r\frac{\partial v_z}{\partial r} + v_r\frac{\partial V_z}{\partial r} + \frac{V_\varphi}{r}\frac{\partial v_z}{\partial \varphi} + \frac{v_\varphi}{r}\frac{\partial V_z}{\partial \varphi} + V_z\frac{\partial v_z}{\partial z} + v_z\frac{\partial V_z}{\partial z} \\
&\qquad = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \varphi^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right), \\
&\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0.
\end{aligned} \tag{16}$$

The definition and algorithm for separation of variables in the non-stationary cylindrical coordinate system are similar to those described in the previous section for the Cartesian coordinate system. Below we show the results.

The most general form of the basic flow is:

$$V_z = A(\xi)T(t) - \frac{c'(t) + zT'(t)}{T(t)}, \qquad V_r = B(\xi)T(t) - r\frac{T'(t)}{T(t)}, \qquad V_\varphi = C(\xi)T(t), \tag{17}$$

where

$$\xi = T(t)r, \qquad \eta = T(t)z + c(t). \tag{18}$$

The forms of the perturbations v_r, v_φ, v_z and p are:

$$\begin{aligned} v_r &= T(t)\exp\left(a\eta + m\varphi + s\int T(t)^2\,\mathrm{d}t\right) f(\xi),\\ v_\varphi &= T(t)\exp\left(a\eta + m\varphi + s\int T(t)^2\,\mathrm{d}t\right) g(\xi),\\ v_z &= T(t)\exp\left(a\eta + m\varphi + s\int T(t)^2\,\mathrm{d}t\right) h(\xi),\\ p &= \rho T(t)^2\exp\left(a\eta + m\varphi + s\int T(t)^2\,\mathrm{d}t\right) \pi(\xi). \end{aligned} \tag{19}$$

The equations with separated variables are

$$\begin{aligned} &f(\xi)(\xi^2 s + \nu - m^2\nu - a^2\xi^2\nu + a\xi^2 A(\xi) + m\xi C(\xi) + \xi^2 B'(\xi)) + 2(m\nu - \xi C(\xi))g(\xi)\\ &\qquad + \xi((-\nu + \xi B(\xi))f'(\xi) + \xi(\pi'(\xi) - \nu f''(\xi))) = 0,\\ &(\xi^2 s + \nu - m^2\nu - a^2\xi^2\nu + a\xi^2 A(\xi) + \xi B(\xi) + m\xi C(\xi))g(\xi) + f(\xi)(-2m\nu + \xi C(\xi)\\ &\qquad + \xi^2 C'(\xi)) + \xi(m\pi(\xi) + (-\nu + \xi B(\xi))g'(\xi) - \xi\nu g''(\xi)) = 0,\\ &(\xi^2 s - m^2\nu - a^2\xi^2\nu + a\xi^2 A(\xi) + m\xi C(\xi))h(\xi) + \xi(a\xi\pi(\xi) + \xi f(\xi)A'(\xi) - \nu h'(\xi)\\ &\qquad + \xi B(\xi)h'(\xi) - \xi\nu h''(\xi)) = 0,\\ &f(\xi) + mg(\xi) + \xi(ah(\xi) + f'(\xi)) = 0. \end{aligned}$$

The restrictions on the forms of the basic flows following from the requirement that they satisfy Navier–Stokes equations lead to the two following cases:

Case I

$$T(t) = \frac{1}{\sqrt{t}}, \qquad B(\xi) = -\frac{3\xi}{4} + \frac{k}{\xi} \tag{20}$$

and the functions $A(\xi)$ and $C(\xi)$ satisfy the equations

$$\begin{aligned} &(4k + 3\xi^2 - 4\nu)A'(\xi) + \xi(-4k + 3\xi^2 + 4\nu)A''(\xi) + 4\xi^2\nu A'''(\xi) = 0,\\ &(4k + 3\xi^2 + 4\nu)C(\xi) + \xi(-4k + 9\xi^2 - 4\nu)C'(\xi)\\ &\qquad + \xi(-4k + 3\xi^2 + 8\nu)C''(\xi) + 4\xi^2\nu C'''(\xi) = 0. \end{aligned}$$

These ODEs can be explicitly solved to give

$$\begin{aligned} A(\xi) &= C_1 + C_2\Gamma\left(\frac{k}{2\nu}, \frac{3\xi^2}{8\nu}\right) + C_3\int \mathrm{e}^{-\frac{3\xi^2}{8\nu}}\xi^{\frac{k}{\nu}-1}\Gamma\left(1 - \frac{k}{2\nu}, -\frac{3\xi^2}{8\nu}\right)\mathrm{d}\xi\\ C(\xi) &= \frac{1}{\xi}\left(C_4 + C_5\Gamma\left(1 + \frac{k}{2\nu}, \frac{3\xi^2}{8\nu}\right) + C_6\int \mathrm{e}^{-\frac{3\xi^2}{8\nu}}\xi^{\frac{k}{\nu}+1}\Gamma\left(-\frac{k}{2\nu}, -\frac{3\xi^2}{8\nu}\right)\mathrm{d}\xi\right) \end{aligned}$$

where $\Gamma(A, Z)$ is the incomplete Gamma function and $C_1, \ldots, C_6$ are arbitrary constants.

Case II

$$T(t) = 1, \qquad B(\xi) = \frac{k}{\xi}. \tag{21}$$

Functions $A(\xi)$ and $C(\xi)$ satisfy the equations

$$(k-\nu)A'(\xi)+\xi(-k+\nu)A''(\xi)+\xi^2\nu A'''(\xi)=0,$$
$$(k+\nu)C(\xi)-\xi(k+\nu)C'(\xi)-\xi^2(k-2\nu)C''(\xi)+r^3\nu C'''(\xi)=0.$$

These ODEs can be explicitly solved in elementary functions. Two cases should be considered: $k\neq 0$ and $k=0$.

Case IIa: $k\neq 0$.

$$A(\xi)=C_1\xi^{\frac{k}{\nu}}+C_2\xi^2+C_3 \qquad C(\xi)=C_4\xi^{\frac{k}{\nu}+1}+C_5\xi^{-1}+C_6\xi.$$

Case IIb: $k=0$.

$$A(\xi)=C_1\ln\xi+C_2\xi^2+C_3 \qquad C(\xi)=C_4\xi\ln\xi+C_5\xi^{-1}+C_6\xi.$$

4. Stability of some unsteady nonparallel flows

In this section, we discuss the fluid dynamics interpretation and stability properties of the exact solutions of the Navier–Stokes equations defined above as basic flows possessing separable stability problems.

4.1. Cartesian coordinates

We will consider the class of solutions in Cartesian coordinates identified in section 2 as case I. We will specify the solutions (6)–(9) by setting $c_1(t)=c_2(t)=c_3(t)=0$ but will use a possibility to enrich the solutions by a shift of the time variable. Making change of variables $t=t'-1/b$, where b is a constant, and omitting primes in what follows, we will have the solution of the Navier–Stokes equations in Cartesian coordinates in the form

$$\begin{aligned}
V_x&=\frac{1}{\sqrt{1-bt}}\left(-\frac{b\xi}{2}+\nu A(\eta)\right),\\
V_y&=\frac{b\eta}{\sqrt{1-bt}},\\
V_z&=\frac{1}{\sqrt{1-bt}}\left(-\frac{b\zeta}{2}+\nu C(\eta)\right),\\
\frac{P}{\rho}&=\frac{1}{1-bt}\left(\frac{b^2}{8}(\xi^2+\zeta^2-8\eta^2)-2\nu^2\left(A_3\xi+C_3\zeta\right)\right),
\end{aligned}\qquad (22)$$

where

$$\xi=\frac{x}{\sqrt{1-bt}},\qquad \eta=\frac{y}{\sqrt{1-bt}},\qquad \zeta=\frac{z}{\sqrt{1-bt}}\qquad (23)$$

and b can be both positive and negative. The functions $A(\eta)$ and $C(\eta)$ are given by

$$\begin{aligned}
A(\eta)&=A_1+A_2\mathrm{erf}\left(\frac{\sqrt{3}\sqrt{-b}}{2\sqrt{\nu}}\eta\right)+A_3\eta^2\,{}_2F_2\left(1,1;\frac{3}{2},2;\frac{3b\eta^2}{4\nu}\right)\\
C(\eta)&=C_1+C_2\mathrm{erf}\left(\frac{\sqrt{3}\sqrt{-b}}{2\sqrt{\nu}}\eta\right)+C_3\eta^2\,{}_2F_2\left(1,1;\frac{3}{2},2;\frac{3b\eta^2}{4\nu}\right),
\end{aligned}\qquad (24)$$

where $\mathrm{erf}(z)$ is the error function, ${}_2F_2(z)$ is the generalized hypergeometric function and A_1, A_2, A_3 and C_1, C_2, C_3 are arbitrary constants.

The correspondingly specified perturbations (10) take the forms

$$
\begin{aligned}
v_x &= (1-bt)^s \exp\left(\mathrm{i}(\alpha\xi+\beta\zeta)\right) f(\eta),\\
v_y &= (1-bt)^s \exp\left(\mathrm{i}\,(\alpha\xi+\beta\zeta)\right) g(\eta),\\
v_z &= (1-bt)^s \exp\left(\mathrm{i}\,(\alpha\xi+\beta\zeta)\right) h(\eta),\\
\frac{p}{\rho} &= (1-bt)^{s-1/2} \exp\left(\mathrm{i}\,(\alpha\xi+\beta\zeta)\right) \pi(\eta).
\end{aligned}
\tag{25}
$$

The equations for the perturbation amplitudes are

$$
\begin{aligned}
&\mathrm{i}\alpha\pi(\eta)+\nu A'(\eta)g(\eta)+\left(-\frac{b}{2}-bs+\nu(\alpha^2+\beta^2)+\mathrm{i}\nu\,(\alpha A(\eta)+\beta C(\eta))\right) f(\eta)\\
&\qquad+\frac{3}{2}b\eta f'(\eta)-\nu f''(\eta)=0,\\
&\pi'(\eta)+(b-bs+\nu(\alpha^2+\beta^2)+\mathrm{i}\nu(\alpha A(\eta)+\beta C(\eta)))g(\eta)+\frac{3}{2}b\eta g'(\eta)-\nu g''(\eta)=0,\\
&\mathrm{i}\beta\pi(\eta)+\nu C'(\eta)g(\eta)+\left(-\frac{b}{2}-bs+\nu(\alpha^2+\beta^2)+\mathrm{i}\nu(\alpha A(\eta)+\beta C(\eta))\right) h(\eta)\\
&\qquad+\frac{3}{2}b\eta h'(\eta)-\nu h''(\eta)=0,\\
&\mathrm{i}(\alpha f(\eta)+\beta h(\eta))+g'(\eta)=0.
\end{aligned}
\tag{26}
$$

The above formulae (22)–(26) remain valid if we introduce the nondimensional variables, with the time scale $1/|b|$ and the correspondingly defined velocity scale. In the dimensionless equations (we will retain the same notation for the nondimensional variables), the parameter b takes one of the two values: $b = 1$ or $b = -1$, and ν is replaced by $1/Re$ where Re is the Reynolds number. (If we mark the dimensional variables with stars, the Reynolds number will be $Re = L^{*2}|b^*|/\nu^*$ where L^* is the length scale.)

Note that the solution (22)–(24) for $b = 1$ undergoes the finite-time 'breakdown' (see, e.g., Duck and Dry (2001), Hall *et al* (1992)) and the 'normal mode' forms (25) are naturally adjusted to the description of the disturbed flow as the breakdown time $t = 1$ is approached.

Next we will consider several two-point boundary value problems, corresponding to different specifications of the basic flow solution (22)–(24), and the results of their analytical and numerical solution allowing assessment of stability of those flows.

4.1.1. Specifications of the basic flow solution. We will consider the solution (22)–(24) for the case of $b = -1$ which allows interpretations corresponding to unsteady flows near stretching (impermeable or permeable) surfaces or the flows that develop within a channel possessing permeable, moving walls. It is worth remarking that the considered flows are essentially nonparallel—the flow fields (22)–(23) include all three velocity components dependent on all coordinates.

(i) Stagnation-point flow towards stretching surface or impingement of two opposite stagnation point flows. These are the simplest basic flows, obtained from (22)–(24) by setting $A(\eta) = C(\eta) = 0$. It is easily seen that they are axially symmetric with respect to the y-axis. The solution can be interpreted as the unsteady stagnation point flow where fluid flowing axially from infinity along the y-axis approaches the rigid surface $y = 0$ stretching radially in its plane (state-state counterparts of this problem were considered in Chiam (1996), Mahapatra and Gupta (2002, 2003), Dauenhauer and Majdalani (2003)). Another possible interpretation is an impingement of two axially symmetric stagnation point flows approaching the plane $y = 0$ from $\pm\infty$ and spreading along it. In both cases, the boundary conditions to equations (26) for perturbations are set at $\eta = 0$ and $\eta = \infty$, as follows:

$$f(0) = 0, g(0) = 0, h(0) = 0, \qquad f, g, h \text{ are bounded at infinity.} \tag{27}$$

(ii) Flow in an expanding channel. In this case the fluid is confined in the space between $y = 1$ and $y = H(t)$. The channel width changes with time as $H = \sqrt{1+t}$ (in the dimensional variables, marked with stars, it is $H^* = H_0^* \sqrt{1 - b^* t^*}$, where $b^* < 0$, and the value H_0^* is used as a length scale for the nondimensional variables while the time scale is $1/|b^*|$). The surfaces stretch radially from the y-axis according to the law $V_r = Kr$ where $V_r = \left(V_x^2 + V_z^2\right)^{1/2}$, $r = (x^2 + z^2)^{1/2}$ and $K = \frac{1}{2}(1+t)^{-1}$. There is an injection of fluid through the porous channel surface at $y = H$, which may be either normal to the surface or oblique—the direction of blowing depends on the values of constants in (24). The blowing velocity varies with time as $V_b = V_0(1+t)^{-1/2}$, where V_0 is a constant. From (24) and the no-slip conditions at the surfaces $y = 1$ and $y = H$ we get for $A(\eta)$, which defines the additional flow due to the pressure gradient along the x-axis, the following:

$$A(\eta) = U_0 \left(\operatorname{erf}\left(\frac{\sqrt{3Re}}{2}\eta \right) - \frac{\eta^2 \operatorname{erf}\left(\frac{\sqrt{3Re}}{2}\right) {}_2F_2\left(1, 1; \frac{3}{2}, 2; \frac{-3\eta^2 Re}{4}\right)}{{}_2F_2\left(1, 1; \frac{3}{2}, 2; \frac{-3Re}{4}\right)} \right) \tag{28}$$

where U_0 is a constant defining the magnitude of the additional flow velocity. For large Re, $A(\eta)$ becomes close to U_0 almost in all the channel.

The boundary conditions for the perturbation amplitudes are set at the surfaces $\eta = 0$ and $\eta = 1$, as follows:

$$f(0) = 0, g(0) = 0, h(0) = 0, \qquad f(1) = 0, g(1) = 0, h(1) = 0. \tag{29}$$

We will make some remarks on the practical and physical relevance of the basic flows considered in the present paper.

The flow and heat transfer occurring within a liquid film on a stretching surface is often encountered in many engineering disciplines. Applications include extrusion process, wire and fibre coating, polymer processing, foodstuff processing, design of various heat exchangers and chemical processing equipment, etc. For example, in a melt-spinning process, the extrudate from the die is generally drawn and simultaneously stretched into a filament or sheet, which is then solidified through rapid quenching or gradual cooling by direct contact with water. Stretching will bring in a unidirectional orientation to the extrudate; consequently, the quality of the final product considerably depends on the flow and heat transfer mechanism. Therefore, the analysis of momentum and thermal transports within a liquid film on a continuously stretching surface is important for gaining some fundamental understanding of such processes. A relevant kind of flow problem is that the flow and heat transfer from a stretching sheet in the infinite surrounding fluid medium. A literature on the subject (see, e.g., the book by Pop and Ingham (2001)) shows considerable research activities in this area.

The basic flows considered in the present paper obey the law for the sheet velocity $U(x, t) = kx/(1 - bt)$. It reflects that the elastic sheet, which is fixed at the origin, is stretched by applying a force in the positive x-direction and the effective stretching rate varies with time as $k/(1 - bt)$. This law for the sheet velocity may seem quite special. Nevertheless, it is commonly accepted in the studies of the momentum and heat transfer over an unsteady stretching surface (see, e.g., Andersson *et al* (2000), Chen (2003), Dandapat *et al* (2003), Devi *et al* (1986), Elbashbeshy and Basid (2004), Lakshmisha *et al* (1988), Takhar and Nath (1996), Usha and Sridharan (1995)). The reasoning behind this is that the flows obtained with this law represent similarity solutions of the Navier–Stokes or/and boundary layer equations (systematic methods for finding such solutions are used, for example, in Ma and Hui (1990), Burde (1995a, 1995b)). It is known that similarity solutions often reflect an asymptotic behaviour of more complicated flows—for example, the similarity solutions of the boundary layer equations serve as asymptotic limit for large distances from the leading edge (see, e.g., Schlichting (1968)). Similarly, one may expect that the flows considered in the present and the above-cited papers might represent as asymptotic limit of the more complicated flows over a stretching surface for large times.

Laminar, incompressible and time-dependent flows that develop within a channel possessing permeable, moving walls have also received considerable attention in the past due to their relevance in a number of engineering applications. Solutions for the physical situations, similar to those considered in the present paper, which pertain to a pipe that exhibits either injection or suction across porous boundaries while undergoing uniform expansion or contraction may be found, for example, in Uchida and Aoki (1977), Dauenhauer and Majdalani (2003).

4.1.2. Solutions of stability problems. We choose as a criterion for stability that the ratio of the magnitude of a perturbation to that of a basic flow decreases with time, which for the solutions (22)–(23) and (25) leads to

$$\mathrm{Re}\left(s+\tfrac{1}{2}\right)<0 \qquad \text{or} \qquad \mathrm{Re}(s)<-\tfrac{1}{2} \tag{30}$$

where $\mathrm{Re}(s)$ denotes a real part of the eigenvalue s (the imaginary part $\mathrm{Im}(s)$, if nonzero, determines the oscillation frequency). In particular, for the decelerating flow ($b=-1$) the meaning of instability implies that even any disturbance is damped ($\mathrm{Re}(s)<0$ for the velocity perturbations and $\mathrm{Re}(s)<1/2$ for the pressure perturbations) yet it may dominate the decelerating flow after sufficient time if $\mathrm{Re}(s)>-1/2$. It is also seen that the condition (30) unifies the stability criterion for the velocity and pressure perturbations.

The eigenvalue problems were solved numerically with the help of the spectral collocation method based on Chebyshev polynomials. Applications of Chebyshev spectral methods to hydrodynamic stability problems have brought out the high degree of accuracy achievable using spectral methods (see, e.g., Khorrami *et al* (1989), Schmid and Henningson (2001) and references therein). Among different possible formulations, a spectral collocation method appears more attractive since a computational algorithm based on that method does not require major modifications from case to case and at the same time the computations are accurate and efficient. Although the spectral collocation technique has been applied to the shear flow stability problems (mainly, to the Orr–Sommerfeld equation), there were no previous applications of the method to the type of problems discussed in this paper. Therefore a Chebyshev collocation matrix algorithm has been constructed for the boundary value problems considered in the paper and the corresponding Fortran computer routine has been designed. Through numerous test cases which, in particular, examined plane Poiseuille flow and Blasius flow, we have shown that the developed algorithm produces accurate global eigenvalues for each case.

For some classes of perturbations, the eigenvalue problems can be solved analytically (see below) which provides an additional, probably the most important, testing of the numerical results.

To specify the stability problem for the basic flows described in section 4.1.1, it should be set $b=-1$ and $\nu=1/Re$ in equations for perturbations (26).

It can be shown that there exists a transformation (similar in a sense to Squire's transformation, Drazin and Reid (1981)) such that the three-dimensional problem defined by equations (26) can be reduced to an equivalent two-dimensional problem. If we let

$$\begin{aligned} &\tilde{\alpha} = (\alpha^2 + \beta^2)^{1/2}, && \tilde{s} = s, \\ &f(\eta) = \frac{1}{\tilde{\alpha}}(\alpha \tilde{f}(\eta) - \beta \tilde{h}(\eta)), && h(\eta) = \frac{1}{\tilde{\alpha}}(\alpha \tilde{h}(\eta) + \beta \tilde{f}(\eta)), \\ &g(\eta) = \tilde{g}(\eta), && \mu(\eta) = \tilde{\mu}(\eta), \\ &A(\eta) = \frac{1}{\tilde{\alpha}}(\alpha \tilde{A}(\eta) - \beta \tilde{C}(\eta)), && C(\eta) = \frac{1}{\tilde{\alpha}}(\alpha \tilde{C}(\eta) + \beta \tilde{A}(\eta)) \end{aligned} \tag{31}$$

then equations (26) can be combined to give the equations in the transformed variables which have exactly the same mathematical form as the original equations with $\beta = 0$ and thus define the equivalent two-dimensional problem. It is sufficient, therefore, to consider only two-dimensional disturbances: for, once the solutions of equations (26) with $\beta = 0$ have been obtained, we can immediately obtain the corresponding solution of the equivalent two-dimensional problem (by a trivial change of notation) and from this, by means of the transformation (31), we can then obtain the solution of the original three-dimensional problem.

Equations (26) with $\beta = 0$ can be reduced to a system of two equations for two functions $g(\eta)$ and $h(\eta)$ of the form

$$\alpha(\alpha b - \alpha bs + \alpha^3\nu + \mathrm{i}\nu(\alpha^2 A(\eta) + A''(\eta)))g(\eta) + \tfrac{3}{2}b\alpha^2\eta g'(\eta) - (b - bs + 2\alpha^2\nu + \mathrm{i}\alpha\nu A(\eta))g''(\eta) - \tfrac{3}{2}b\eta g'''(\eta) + \nu g^{(\mathrm{IV})}(\eta) = 0 \tag{32}$$

$$\nu C'(\eta)g(\eta) + \left(-\tfrac{1}{2}b - bs + \alpha^2\nu + \mathrm{i}\alpha\nu A(\eta)\right)h(\eta) + \tfrac{3}{2}b\eta h'(\eta) - \nu h''(\eta) = 0. \tag{33}$$

It is seen that for $C(\eta) = 0$ the system of equations (32) and (33) decouples into two separate equations for $g(\eta)$ and $h(\eta)$. Thus, in this case two separate branches exist, the first of which corresponds to the disturbances with one z-component of the velocity vector changing with x and y, while the second branch corresponds to the two-dimensional disturbances with velocity vector lying in the (x, y) plane and not dependent on z.

In the case where both $A(\eta) = 0$ and $C(\eta) = 0$ equations (32) and (33) can be solved in quadratures in terms of confluent hypergeometric functions. Each of equations can be reduced to Kummer's equation (Abramowitz and Stegun 1965)

$$Z\frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} + (B - Z)\frac{\mathrm{d}W}{\mathrm{d}Z} - QW = 0, \tag{34}$$

where

$$Z = -\frac{3\eta^2}{4\nu}, \qquad B = \frac{1}{2} \qquad \text{for both equations}, \tag{35}$$

$$W = g''(\eta) - \alpha^2 g(\eta), \qquad Q = -\frac{1}{3}(-1 + s + \alpha^2\nu) \qquad \text{for equation (32)}, \tag{36}$$

$$W = h(\eta), \qquad Q = -\frac{1}{3}\left(\frac{1}{2} + s + \alpha^2\nu\right) \qquad \text{for equation (33)}. \tag{37}$$

A general solution of Kummer's equation (34) can be taken in the form

$$W = C_1 M(Q, B, Z) + C_2 Z^{1-B} M(1 + Q - B, 2 - B, Z), \tag{38}$$

where $M(Q, B, Z)$ is the Kummer confluent hypergeometric function and C_1 and C_2 are arbitrary constants.

Let us consider the perturbation mode with one z-component of the velocity vector changing with x and y which corresponds to $g(\eta) = f(\eta) = 0$ and $h(\eta)$ determined by (34), (35), (37) and (38). The boundary condition $h(0) = 0$ requires $C_1 = 0$ since $M(Q, B, 0) = 1$ (Abramowitz and Stegun 1965) so that the solution for $h(\eta)$ satisfying equation (33) and the boundary condition at $\eta = 0$ takes the form

$$h(\eta) = \eta M\left(\frac{1}{3}(1 - s - \alpha^2\nu), \frac{3}{2}, -\frac{3\eta^2}{4\nu}\right). \tag{39}$$

Using (39) in the second boundary condition leads to a transcendental equation determining eigenvalues s for each α and $\nu = 1/Re$ (here Re is the Reynolds number).

In the cases where the second boundary condition is set at infinity (for example, a stagnation point flow to a stretching surface), one can use a limiting form of Kummer's function (Abramowitz and Stegun 1965),

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}(1 + O(|z|^{-1})) \qquad (\mathrm{Re}(z) < 0). \tag{40}$$

Then the condition that the perturbations $h(\eta)$ are bounded at infinity applied to (39) with the use of (40) gives a continuous real spectrum defined by

$$s \leqslant -\frac{\alpha^2}{Re} - \frac{1}{2}. \tag{41}$$

Comparing this with (30) we can conclude that the axially symmetric flow defined by (22)–(23) with $A(\eta) = C(\eta) = 0$ in the semi-infinite domain is stable with respect to the perturbations having only z-velocity component dependent on x and y.

Similar arguments can be applied to the second perturbation mode (the two-dimensional disturbances with velocity vector lying in the (x, y) plane and not dependent on z), which corresponds to $h(\eta) = 0$, $g(\eta)$ determined by (34), (35), (36) and (38) and $f(\eta)$ defined by the last equation of (26) with $\beta = 0$. Using the limiting form of Kummer's function (40) together with the condition that the perturbations are bounded at infinity yields

$$s \leqslant 1 - \frac{\alpha^2}{Re} \qquad \text{or} \qquad s + \frac{1}{2} \leqslant \frac{3}{2} - \frac{\alpha^2}{Re}. \tag{42}$$

Applying criterion (30) we can see that the axially symmetric flow defined by (22)–(23) with $A(\eta) = C(\eta) = 0$ in the semi-infinite domain may be unstable with respect to the two-dimensional perturbations if $3/2 - \alpha^2/Re > 0$. Thus, the instability region is determined by

$$\alpha < \sqrt{\frac{3}{2}}Re^{1/2}. \tag{43}$$

In the case of a channel flow, when the second boundary condition for each component is set at the wall $\eta = 1$, the eigenvalues form a discrete spectrum. For the first perturbation mode, for example, it is found from the transcendental equation

$$M\left(\frac{1}{3}\left(1 - s - \frac{\alpha^2}{Re}\right), \frac{3}{2}, -\frac{3Re}{4}\right) = 0 \tag{44}$$

obtained by applying the condition $h(1) = 0$ to (39). Note, however, that analysis of stability of the channel flow, in general, should include the cases when the basic flow, in addition to an axially symmetric part, includes a parallel flow with $A(\eta)$ and $C(\eta)$ defined as in (28). The solution of the corresponding eigenvalue problem can be obtained only with the use of a numerical method. We do not consider those results not to overload the presentation—similar results for a flow within a cylinder are discussed below.

Nevertheless, it should be emphasized that the analytical results like (44) provide a very useful test for numerical methods of solution of eigenvalue problems and for methods used in

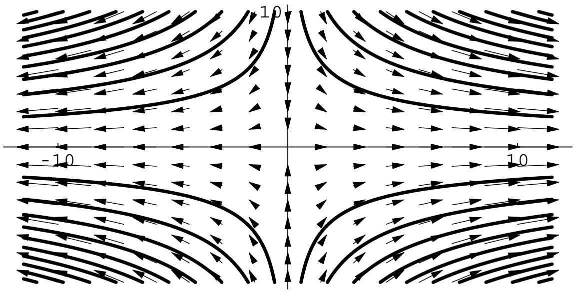

Figure 1. Unsteady axially symmetric stagnation point flow: $b = -1$.

the hydrodynamic stability theory, in general. Testing the spectral collocation method used in the present work with (44) showed that the method can reproduce the analytical results with high accuracy.

4.2. Cylindrical coordinates

Here we will consider the class of solutions of the Navier–Stokes equations in cylindrical coordinates identified in section 3 as case I, which is similar in many features to the class of solutions in Cartesian coordinates considered in the previous section. Therefore, we will not write out the solutions for the basic flow and perturbations and will not present detailed formulations of the stability problems. We will only state that the solution in cylindrical coordinates can be enriched by a shift of the time variable, like as it has been done for the solution in Cartesian coordinates. Again, all the formulae remain valid if we introduce the nondimensional variables, with the time scale $1/|b|$ and the correspondingly defined velocity scale. Then, in the dimensionless equations, the parameter b takes one of the two values: $b = 1$ and $b = -1$, and ν is replaced by $1/Re$ where Re is the Reynolds number. We will again restrict ourselves to the case of $b = -1$.

4.2.1. Interpretations of the basic flow solution. The basic flow solution in cylindrical coordinates permits interpretations similar to those considered above for the solution in Cartesian coordinates. However, the cylindrical geometry and presence of the additional free parameters k (see equation (20)) allows one to find more problem formulations and enrich the problem definitions. The basic flow might be again an unsteady axially symmetrical stagnation-point type flow, with the flow velocity decreasing with time as $(1 + t)^{-1}$, but, as distinct from the flows considered in the previous section; here fluid flows radially from infinity approaching the axis and spreading along it (figure 1).

The basic flow might be an unsteady flow inside an expanding stretching cylinder, which may also rotate, and there is an injection of fluid through the porous pipe surface. Such a flow without rotation is shown in figure 2 where the meaning of U_0 is similar to that in (28)—it is a constant defining a relative magnitude of the part of the flow caused by the axial pressure gradient.

It is seen from figures 1 and 2 that the considered flows are essentially nonparallel. Other basic flow specifications include a flow outside an expanding stretching cylinder (both impermeable and permeable), which can also rotate, and flows in the gap between concentric cylinders—here again the cylinders may be permeable or not permeable and may rotate or not rotate.

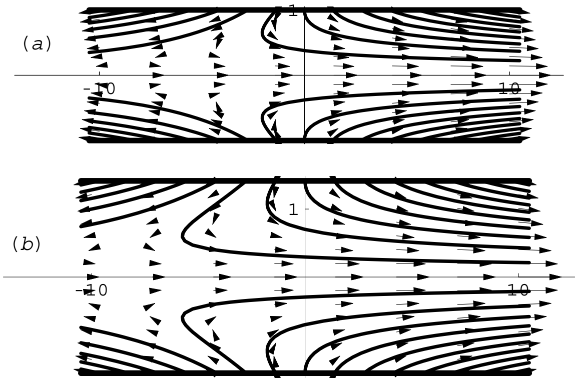

Figure 2. Flow inside an expanding porous cylinder for $Re = 100$ and $U_0 = 5$ at different time moments: (*a*) $t = 0$, (*b*) $t = 1$.

4.2.2. Solutions of stability problems. The time dependence of the basic flow and perturbation solutions in cylindrical coordinates is similar to that of the solutions (22)–(23) and (25). Therefore a criterion for stability coincides with that given by (30).

There is an important point in which the stability problems in cylindrical coordinates differ from those in Cartesian coordinates: a transformation, similar to Squire's transformation, which reduces the three-dimensional perturbation problem to an equivalent two-dimensional problem, does not exist. Therefore, in general, one has to consider the three-dimensional perturbations to assess the flow stability. Nevertheless, as usual in the stability analysis (see, e.g., Drazin and Reid (1981), Schmid and Henningson (2001)), it is useful to study stability properties with respect to the two-dimensional or axisymmetric perturbations since it enables one to show that the flow is susceptible (or not susceptible) to a special kind of instability. For some of the flows considered here, the eigenvalue problems restricted to the two-dimensional or axisymmetric perturbations are amenable to analytical solution which allows one to obtain exact results similar to those given by (39) and (41)–(44)).

Below we present the results of numerical solution of the eigenvalue problems for the most general three-dimensional perturbations. We will restrict ourselves to considering the unsteady nonparallel flows developing within expanding pipe in order not to overload the presentation.

First, the analysis shows that the flow within not rotating cylinder and in the absence of the axial pressure gradient is stable—one can see this in figure 3 where contours of constant (relative) growth rate $S = \mathrm{Re}(s) + 1/2$ are shown on the plane (Re, α). Since it is not $\mathrm{Re}(s)$ but $S = \mathrm{Re}(s)+1/2$, whose values are shown in this and subsequent figures, unstable solutions correspond to $S > 0$ and stable ones to $S < 0$ while the neutral curve (if exists) is the contour of $S = 0$. In figure 3, unstable solutions do not exist in all the parameter space. All the eigenvalues are real so that the disturbances decay monotonically.

The contours of positive values of S appear in figure 4 which corresponds to the situation when the basic flow includes the part due to the axial pressure gradient ($U_0 \neq 0$). The neutral curve $S = 0$ separates the regions of stability and instability. It is seen that for any Reynolds number larger than some critical value Re_* (for $U_0 = 30$, $Re_* \approx 120$) there exist a range of wave numbers α corresponding to unstable solutions. Thus, the flow including the part due to the axial pressure gradient is unstable for $Re > Re_*$. The critical Reynolds number Re_* decreases while U_0 increases. It is worth remarking that, despite the fact that the basic flow diverges from the 'centre' $x = 0$, the unstable modes defined by (25) are periodic in the axial

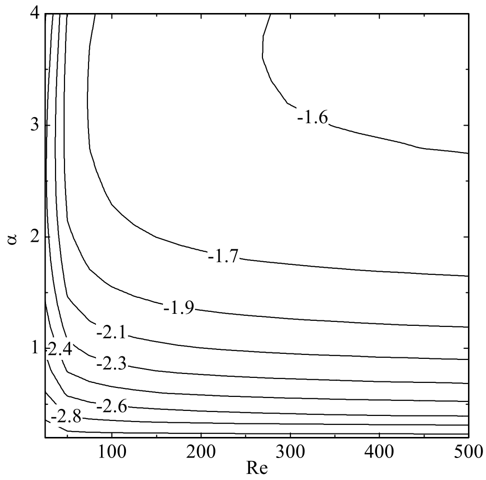

Figure 3. Contours of constant growth rate S; $U_0 = 0$ and $n = 2$.

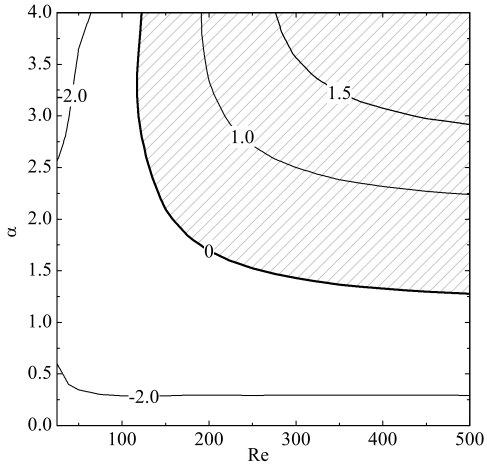

Figure 4. Neutral curve and contours of constant growth rate S for $U_0 = 30$ and $n = 2$. The shaded area represents the region in parameter space where unstable solutions exist.

direction with the wavelength increasing with time. Variation of the perturbation amplitudes in the radial direction is shown in figure 5.

The unsteady flow within expanding cylinder may be unstable if $U_0 = 0$ but the cylinder rotates. As distinct from the case of not rotating cylinder, in this case the instability region is close to the horizontal axes so that the unstable modes may correspond to any small wave number α while there is a maximal wave number for instability for any $Re > Re_*$.

5. Concluding remarks

In the present paper, we have applied the so-called direct approach to separate variables in the linear stability equations. As the result, we have defined several classes of the *exact solutions of*

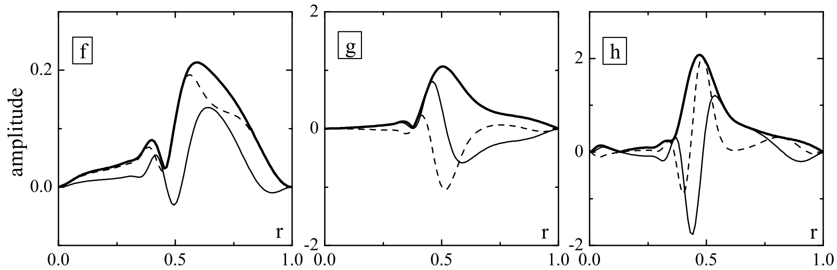

Figure 5. Eigenfunctions for $U_0 = 30, n = 2, \alpha = 2, Re = 200$; f, g and h are respectively the amplitudes of the radial, azimuthal and axial components of perturbations. The thick solid line represents the absolute value of f or g or h; the thin solid and dashed lines represent respectively the real and imaginary parts.

the Navier–Stokes equations, for which the linear stability problems are *exactly solvable*, and determined the corresponding forms of perturbations and equations with separated variables. It should be noted that this is the first success in the rigorous stability analysis for a nonparallel unsteady flow that reduces the exact disturbance equations to an eigenvalue problem of ordinary differential equations. This should help in furthering current understanding of the nonparallel flow instability physics and can provide a necessary foundation for many approximate approaches used so far. The results from exactly separable stability problems can be used for testing various assumptions and simplifications on which those theories are based.

We extended the analysis to solve numerically the eigenvalue problems for some flows. The results include determining the growth rates, instability regions in parameter space and critical Reynolds numbers. Thus, we presented the nonparallel unsteady flow stability problems solved via separation of variables. In addition, we found that, for some classes of perturbations, the eigenvalue problems can be solved analytically. Those unique examples of exact (explicit) solution of the nonparallel unsteady flow stability problems provide a very useful test for methods used in the hydrodynamic stability theory.

The general forms of the basic flows, which have been obtained from the only requirement of separability of the corresponding stability problem, are richer than those remaining after specification to the exact solutions of the Navier–Stokes equations. Thus, using the approach accepted in many stability studies, where the form of the basic flow is chosen quite freely to approximate the physical situation of interest, we could considerably enrich the list of relevant flows.

It should be also noted that in some studies the stability analysis is restricted to the two-dimensional perturbations even though an analogue of the Squire theorem cannot be proved (see, e.g., Griffond and Casalis (2001), Joslin (1996)). Such analysis cannot be complete since it enables one only to show that the flow is susceptible to instability, if the growing perturbations exist, but a stability of the flow cannot be proved. We found that application of our algorithm of separation of variables with a specification of the disturbance field to the two-dimensional perturbations may lead to new possibilities (other forms of basic flow and perturbation fields), but we do not consider these results here.

Acknowledgments

This research was supported by the Israel Science Foundation (grant no. 117/03). The research of AZ and ISN was also supported by the postdoctoral fellowships at the Jacob Blaustein

Institute for Desert Research of the Ben-Gurion University of the Negev. AZ acknowledges support from NATO Reintegration Grant NUKR.RIG 982038 and grant of the President of Ukraine for young scientists GP/F11/0061.

References

Abramowitz M and Stegun I 1965 *Handbook of Mathematical Functions* (New York: Dover)

Andersson H I, Aarseth J B and Dandapat B S 2000 Heat transfer in a liquid film on an unsteady stretching surface *Int. J. Heat Mass Transfer* **43** 69–74

Burde G I 1995a The construction of special explicit solutions of the boundary-layer equations. Unsteady flows *Q. J. Mech. Appl. Math.* **48** 611–33

Burde G I 1995b Nonsteady stagnation-point flows over permeable surfaces: explicit solutions of the Navier-Stokes equations *J. Fluid Eng. Trans. ASME* **117** 189–91

Chen C H 2003 Heat transfer in a power-law fluid film over a unsteady stretching sheet *Heat Mass Transfer* **39** 791–796

Chiam T C 1996 Heat transfer with variable conductivity in a stagnation point flow towards a stretching sheet *Int. Commun. Heat Mass Transfer* **23** 239–48

Dandapat B S, Santra B and Andersson H I 2003 Thermocapillarity in a liquid film on an unsteady stretching surface *Int. J. Heat Mass Transfer* **46** 3009–15

Dauenhauer E C and Majdalani J 2003 Exact self-similarity solution of the Navier–Stokes equations for a porous channel with orthogonally moving walls *Phys. Fluids* **15** 1485–94

Devi C D S, Takhar H S and Nath G 1986 Unsteady, three-dimensional, boundary-layer flow due to a stretching surface *Int. J. Heat Mass Transfer* **39** 1998–9

Drazin P G and Reid W H 1981 *Hydrodynamic Stability* (Cambridge: Cambridge University Press)

Duck P and Dry S L 2001 On a class of unsteady, non-parallel, three-dimensional disturbances to boundary-layer flows *J. Fluid Mech.* **441** 31–65

Elbashbeshy E M A and Basid M A A 2004 Heat transfer over an unsteady stretching surface *Heat Mass Transfer* **41** 1–4

Fazel H and Konzelmann U 1990 Non-parallel stability of a flat plate boundary layer using the complete Navier-Stokes equations *J. Fluid Mech.* **221** 311–47

Govindarajan R and Narasimha R 1999 Low-order parabolic theory for 2D boundary-layer stability *Phys. Fluids* **11** 1449–58

Griffond J and Casalis G 2001 On the nonparallel stability of the injection induced two-dimensional Taylor flow *Phys. Fluids* **13** 1635–44

Hall P, Balakumar P and Papageorgiu D 1992 On a class of unsteady three-dimensional Navier-Stokes solutions relevant to rotating disk flows: threshold amplitudes and finite-time singularities *J. Fluid Mech.* **238** 297–323

Herbert Th 1997 Parabolized stability equations *Annu. Rev. Fluid. Mech.* **29** 245–83

Joslin R D 1996 Simulation of nonlinear instabilities in an attachment-line boundary layer *Fluid Dyn. Res.* **18** 81–97

Khorrami M, Malik M R and Ash R L 1989 Application of spectral collocation techniques to the stability of swirling flows *J. Comp. Phys.* **81** 206–29

Lakshmisha K N, Venkateswaran S and Nath G 1988 Three-dimensional unsteady flow with heat and mass transfer over a continuous stretching surface *J. Heat Transfer Trans. ASME* **110** 590–5

Ma P K H and Hui W H 1990 Similarity solutions of the two-dimensional unsteady boundary layer equations *J. Fluid Mech.* **216** 537–59

Mahapatra T R and Gupta A S 2002 Heat transfer in stagnation-point flow towards a stretching sheet *Heat Mass Transfer* **38** 517–21

Mahapatra T R and Gupta A S 2003 Stagnation-point flow towards a stretching surface *Can. J. Chem. Eng.* **81** 258–63

Pop I and Ingham D B 2001 *Convective Heat Transfer: Mathematical and Computational Modelling of Viscous Fluids and Porous Medis* (Oxford: Pergamon)

Reed H L, Saric W S and Arnal D 1996 Linear stability theory applied to boundary layers *Annu. Rev. Fluid Mech.* **28** 389–428

Saric W S, Reed H L and White E B 2003 Stability and transition of three-dimensional boundary layers *Ann. Rev. Fluid. Mech.* **35** 413–40

Schlichting H 1968 *Boundary Layer Theory* 6th edn (New York: McGraw-Hill)

Schmid P J and Henningson D S 2001 *Stability and Transition in Shear Flows* (New York: Springer)

Takhar H S and Nath G 1996 Unsteady three-dimensional flow due to a stretching flat surface *Mech. Res. Commun.* **23** 325–33

Uchida S and Aoki H 1977 Unsteady flows in a semi-infinite contracting or expanding pipe *J. Fluid Mech.* **82** 371–87

Usha R and Sridharan R 1995 The axisymmetric motion of a liquid film on am unsteady stretching surface *J. Fluid Eng. Trans. ASME* **117** 81–5

Zhalij A 1999 On separable Fokker–Planck equations with a constant diagonal diffusion matrix *J. Phys. A: Math. Gen.* **32** 7393–404

Zhalij A 2002 On separable Pauli equations *J. Math. Phys.* **43** 1365–89

Zhdanov R and Zhalij A 1999a On separable Schrödinger equations *J. Math. Phys.* **40** 6319–38

Zhdanov R and Zhalij A 1999b Separation of variables in the Kramers equation *J. Phys. A: Math. Gen.* **32** 3851–63